Exam **Discrete Mathematics (resit), February 1st 2012, 09:00h-12:00h**
Bachelor program Knowledge Engineering
Bouillonstraat 8-10, room 0.015

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments. Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$. Good luck!

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1. **(15 points)** Use induction to prove the following statements:

   (a) For all integers $n \geq 0$,

   $$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

   (b) For all integers $n \geq 0$, $\quad 5^n - 1$ is divisible by 4.

To finish

2. **(10 points)** Fill in the truth table for the following logical proposition:

   - $((p \Rightarrow q) \wedge r) \Leftrightarrow (\neg r \wedge (p \vee (q \wedge r)))$

later

3. **(15 points)** Prove or disprove:

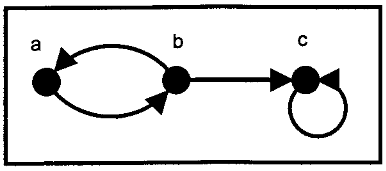
   - For all sets A, B, and C, if $(C \setminus (A \cap B)) = (A \cap C) \cup (B \cap C)$, then $C \subseteq B \cup A$.

4. **(15 points)** Prove or disprove the following statements.

   (a) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})(\forall z \in \mathbb{Z})(x + y \leq z)$

   (b) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{N})(\forall z \in \mathbb{N})(y^x \leq z)$

5. **(15 points)** Recall that a relation diagram allows us to visualise a relation. For example, if $R = \{(a, b), (b, a), (b, c), (c, c)\}$ is a relation on $A = \{a, b, c\}$, the relation diagram looks like this:



   In the course we studied the following three properties of a relation: *reflexive*, *transitive*, and *symmetric*. There are $2^3 = 8$ different ways of combining these properties. For example, a relation might have all three properties, or none of them, or some but not others. For each of the 8 possible combinations, construct a relation on A which has exactly these properties, and draw its relation diagram.

6. **(10 points)** Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be functions defined as follows.

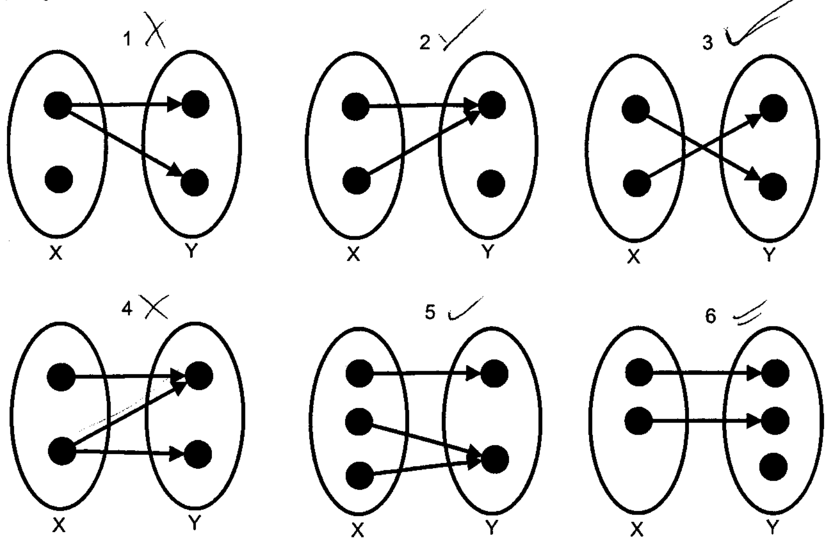
   $$f(x) = \begin{cases} x & \text{if } 1 \leq x \leq 10 \\ 2x & \text{if } x > 10 \end{cases}$$

   $$g(x) = \begin{cases} x & \text{if } 1 \leq x \leq 5 \\ x^2 & \text{if } 6 \leq x \leq 30 \\ x + 1 & \text{if } x > 30 \end{cases}$$

   Give, in the same style as the definitions for $f(x)$ and $g(x)$, the definition of the function $(g \circ f)(x)$.

7. **(15 points)**

(a) For each of the six diagrams below, say whether it shows a function $f : X \to Y$. In each case, if it is a function, say whether the function is injective, surjective, both or neither.



(b) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined as follows, where ${}^+\sqrt{x}$ refers to the positive square root of x:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 5 \\ {}^+\sqrt{(x-1)} + 23 & \text{if } x > 5 \end{cases}$$

Prove that f is invertible. Then give f^{-1} and verify that it is indeed the inverse of f.

8. **(5 points)** Let B_F be the set of all finite length binary strings. Is B_F countable or uncountable? Use an appropriate argument to motivate your answer.