Exam **Discrete Mathematics, October 28th 2011, 09:00h-12:00h**
Bachelor program Knowledge Engineering
Bouillonstraat 8-10, rooms 0.009, 0.015 and 2.015

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Good luck!

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1. **(15 points)** Use induction to prove the following statements: 15

   (a) For all integers $n \geq 1$, $\quad 2 + 4 + 6 + \ldots + 2n = n^2 + n$.

   (b) For all integers $n \geq 0$, $\quad 2^{2n} - 1$ is divisible by 3.

2. **(10 points)** Fill in the truth table for the following logical proposition: 10

   - $((p \vee r) \Rightarrow q) \Leftrightarrow ((\neg q \wedge p) \Rightarrow r)$

3. **(15 points)** Prove or disprove:

   - For all sets A, B, and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$, then $A \subseteq B$. 10

4. **(15 points)** Let $A = \{0, 1, 2, 3, 4\}$. Prove or disprove: 5

   (a) $(\forall x \in A)(\exists y \in A)(x \leq (y+1)^2 \leq x + 2)$

   (b) $(\exists x \in A)(\forall y \in A)(y \leq x - 1)$

5. **(15 points)** Let R be a relation on $\mathbb{N}$ defined as follows: xRy means "$x + 3y$ is even". Prove or disprove that R is an equivalence relation. If it is an equivalence relation, describe its equivalence classes. 5

6. **(10 points)** Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be functions defined as follows.

$$f(x) = \begin{cases} x^2 & \text{if } 1 \leq x \leq 10 \\ 2x + 1 & \text{if } x > 10 \end{cases}$$

$$g(x) = \begin{cases} x + 2 & \text{if } 1 \leq x \leq 50 \\ 3x & \text{if } x > 50 \end{cases}$$

   Give, in the same style as the definitions for $f(x)$ and $g(x)$, the definition of the function $(g \circ f)(x)$.

7. **(15 points)** Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined as follows, where ${}^+\sqrt{x}$ refers to the positive square root of x:

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x < 4 \\ {}^+\sqrt{x} + 10 & \text{if } 4 \leq x < 16 \\ (x/8) + 12 & \text{if } x \geq 16 \end{cases}$$

   Prove that f is invertible. Then give f^{-1} and verify that it is indeed the inverse of f.

8. **(5 points)** Let $\mathbb{Q}$ be the set of rational numbers. Is $(\mathbb{Q} \cap \mathbb{R}) \times (\mathbb{Z} \cap \mathbb{R})$ countable or uncountable? Use an appropriate argument to motivate your answer.